LITERATURE CITED

1. A. H. Whitehead, Jr., J. N. Hefner, and D. M. Rao, "Lee-surface vortex effects over configurations in hypersonic flow," AIAA Paper No. 72-77 (1972).
2. D. M. Rao and A. H. Whitehead, Jr., "Lee-side vortices on delta wings at hypersonic speeds," AIAA J., 10, No. 11 (1972).
3. E. J. Cross, Jr., and W. L. Hankey, "Investigation of leeward side of a delta wing at hypersonic speeds," AIAA Paper No. 68-675 (1968).
4. M. D. Ladyzhenskii, "On spatial hypersonic flow around thin wings," Prikl. Mat. Mekh., 28, No. 5 (1964).
5. I. G. Kozlova and V. V. Mikhailov, "On strong viscous interaction on triangular and sliding wings," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1970).
6. V. Ya. Neiland, "On the theory of hypersonic flow interaction with the boundary layer for two- and three-dimensional separation flows. Pt. 2. Two-dimensional flows and a delta wing," Uch. Zap. TsAGI, 5,No. 3 (1974).
7. G. N. Dudin, "On a boundary-layer computation on a triangular plate in the strong viscous interaction regime," Uch. Zap., Ts. AGI, 9, No. 5 (1978).
8. G. S. Bluford, Jr., "Numerical solution of the supersonic and hypersonic viscous flow around thin delta wings," AIAA Paper No. 78-1136 (1978).
9. W. D. Hayes and R. F. Probstein (eds.) Hypersonic Flow Theory, Academic Press (1967).
10. V. Ya. Neiland, "On the theory of hypersonic flow interaction with the boundary layer for two- and three-dimensional separation flows. Pt. 1. Three-dimensional flows," Uch. Zap., TsAGI, 5, No. 2 (1974).
11. Yu. D. Shevelev, Three-Dimensional Problems of Laminar Boundary Layer Theory [in Russian], Nauka, Moscow (1977).

STABILITY OF A PLANE CRYSTALLIZATION FRONT MOVING AT CONSTANT VELOCITY

L. G. Badratinova

UDC 532.78:536.421.4

1. In an x, y, z coordinate system coupled to a plane unperturbed front (the x axis is directed into the melt and the y, z axes are along the interfacial surface), the crystallization process of a dilute binary alloy is described by the equations

$$\begin{gathered}\text{for} \quad x > f(y, z, t) \quad \partial T_1/\partial t + \mathbf{v}_1 \cdot \nabla T_1 = \chi_1 \Delta T_1, \\ \partial c_1/\partial t + \mathbf{v}_1 \cdot \nabla c_1 = D \Delta c_1, \ \nabla \cdot \mathbf{v}_1 = 0, \\ \partial \mathbf{v}_1/\partial t + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = \nu \Delta \mathbf{v}_1 - \nabla p/\rho_1 + \mathbf{g}, \ \mathbf{g} = (-g, 0, 0);\end{gathered} \tag{1.1}$$

$$\text{for} \quad x < f(y, z, t) \quad \partial T_2/\partial t - V_2(\partial T_2/\partial x) = \chi_2 \Delta T_2 \tag{1.2}$$

with local phase equilibrium conditions [1]

$$x = f(y, z, t) \quad T_1 = T_2 = mc_1 + T_0 + T_0 \gamma K, \tag{1.3}$$

no tangential component of the melt velocity on the front, and continuity of the energy and mass fluxes [2] of both melt components during passage through the interface x = f(y, z, t):

$$\begin{gathered}(\varkappa_2 \nabla T_2 - \varkappa_1 \nabla T_1)\mathbf{n} = -\rho_1 \Lambda (\mathbf{v}_1 - \mathbf{U})\mathbf{n}, \\ \mathbf{v}_1 \cdot \boldsymbol{\tau} = 0, \ \rho_1 (\mathbf{v}_1 - \mathbf{U})\mathbf{n} = \rho_2 (\mathbf{v}_2 - \mathbf{U})\mathbf{n}, \\ D\rho_1 \rho_2^{-1} \nabla c_1 \mathbf{n} = (1 - k) c_1 (\mathbf{v}_2 - \mathbf{U})\mathbf{n}.\end{gathered} \tag{1.4}$$

Here $\mathbf{v}_1$, p, and c_1 are the melt velocity, pressure, and impurity concentration (measured in weight fractions), T_j (j = 1, 2) are the temperatures of the medium. The subscript

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 113-120, May-June, 1983. Original article submitted May 17, 1982.

 0021-8944/83/2403-0388$07.50

1 characterizes the liquid phase and the subscript 2 the solid. The density of the medium and the thermal diffusivities and conductivities are denoted by ρ_j, χ_j, $\varkappa_j$, and D are the kinematic viscosity and impurity diffusion coefficients, and $V_2 = \text{const}$ is the crystal velocity. The dependence of the crystallization temperature on the impurity concentration at the front (m is the slope of the liquidus line) as well as on the curvature K of the interfacial surface is taken into account in condition (1.3); γ is the surface tension coefficient, T_0 is the melting point on the plane front in the absence of impurities, Λ is the latent heat of crystallization, k is the equilibrium coefficient of impurity distribution, and n, τ, and U are the normal and tangential vectors to the phase transition boundary and the displacement velocity of this boundary.

It is assumed that the melt occupies the whole half-space $x > f(y, z, t)$, and a given temperature is maintained in the solid medium at a certain fixed distance H from the unperturbed front. The following are appended to conditions (1.3) and (1.4):

$$\begin{aligned} &\text{for } x = \infty \; c_1 = c_\infty, \; T_1 = T_\infty, \; \mathbf{v}_1 = (-V_1, 0, 0), \\ &\text{for } x = -H \; T_2 = T_{20}[1 - \exp(-V_2 H/\chi_2)] + T_0 + mc_\infty/k, \end{aligned} \tag{1.5}$$

where $V_1 = \rho_2 V_2/\rho_1$; $T_{20} = (c_{V_1}T_1 + \Lambda)/c_{V_2}$; $T_{10} = T_\infty - T_0 - mc_\infty/k$; c_{V_j} are the specific heats of the media for constant volume.

2. Problem (1.1)-(1.5) has a one-dimensional stationary solution

$$\begin{aligned} &f^0 \equiv 0, \quad \mathbf{v}_1^0 = (-V_1, 0, 0), \quad c_1^0 = c_\infty + c_\infty(k^{-1} - 1)\exp(-V_1 x/D), \\ &p^0 = -\rho_1 g x + \text{const}, \quad T_j^0 = T_{j0}[1 - \exp(-V_j x/\chi_j)] + T_0 + mc_\infty/k. \end{aligned} \tag{2.1}$$

For simplicity, the stability is investigated in the plane case. (All the results remain valid even for three-dimensional flow.) We linearize the system (1.1)-(1.5) in the neighborhood of the solution (2.1) and we introduce the stream function $\psi_1(x, y, t)$, by setting $\mathbf{v}_1 = (\partial\psi_1/\partial y, -\partial\psi_1/\partial x)$. We eliminate the pressure p and reduce the system to dimensionless form by selecting the following units of measurement as independent: the length χ_1/V_1, the time χ_1/V_1^2, the temperature Λ/C_{V_1}, and the concentration c_∞/k. Let us separate the time t and the variable y by setting

$$\{f(y, t), c_1(x, y, t), T_j(x, y, t), \psi_1(x, y, t)\} = \{ia, ic(x), i\theta_j(x), \psi(x)\}\exp(\mu t + i\omega y) \quad (j = 1, 2).$$

Consequently, we obtain a system of ordinary differential equations for the amplitudes of the normal perturbations

$$\begin{aligned} \text{for } x > 0 \quad &(L - \mu)\theta_1 + \theta_1' = \omega G_1 \psi \exp(-x), \\ &(L - \mu D_*)c + D_* c' = \omega D_* G_c \psi \exp(-D_* x), \\ &L[(L - \mu \mathrm{Pr}_*)\psi + \mathrm{Pr}_* \psi'] = 0; \end{aligned} \tag{2.2}$$

$$\text{for } x < 0 \quad (L - \mu\chi_*)\theta_2 + \rho_*\chi_*\theta_2' = 0; \tag{2.3}$$

with the boundary conditions

$$\begin{aligned} &\text{for } x = 0 \; \psi' = 0, \; \omega\psi = -\varepsilon\mu a, \\ &\theta_1 + G_1 a = \theta_2 + G_2 a = m_0(c + G_c a) - \gamma_0\omega^2 a, \\ &\varkappa_*^{-1}(\theta_2' - \rho_*\chi_* G_2 a) - \theta_1' + G_1 a = \rho_*^{-1}\mu a, \\ &D_*^{-1}c' - kG_c a + (1 - k)c = \mu\rho_*^{-1}(k - 1)a; \\ &\text{for } x = \infty \; \theta_1 = c = 0, \; \psi = \psi' = 0; \\ &\text{for } x = -h \; \theta_2 = 0. \end{aligned} \tag{2.4}$$

Here $L = (d^2/dx^2) - \omega^2$, the prime denotes differentiation with respect to x, $\mathrm{pr}_* = \chi/\nu$ is the reciprocal Prandtl number, $D_* = \chi_1/D$; $\chi_* = \chi_1/\chi_2$; $\varkappa_* = \varkappa_1/\varkappa_2$; $\rho_* = \rho_1/\rho_2$ $\varepsilon = (\rho_2 - \rho_1)/\rho_1$; the quantities $G_c(k - 1)D_*$, $G_1 = c_{V_1}T_{10}/\Lambda$ and $G_2 = \varkappa_*(1 + G_1)$ are dimensionless gradients of the impurity concentration and of the temperatures in the melt and in the crystal, and $h = HV_1/\chi_1$; $m_0 = mc_\infty c_{V_1}/k\Lambda$; $\gamma_0 = \gamma T_0 c_{V_1} V_1/\chi_1\Lambda$.

For $G_1 < 0$ the temperature T_1^0 in the melt is below the temperature $T_0 + mc_\infty/k$ on the crystallization front. We shall allow the possibility of fluid supercooling (see [3, 4]) and consider the problem for $G_1 \geqslant -1$. If $G_1 = -1$, then the gradient $G_2 = 0$ and the temperature in the crystal is $T_{20} = T_0 + mc_\infty/k$.

If all the eigenvalues μ of the problem (2.2)-(2.4) have a negative real part, then solution (2.1) is stable. If at least one perturbation is in the spectrum for which $\mathrm{Re}\,\mu > 0$, then solution (2.1) is unstable relative to this perturbation.

3. Let $D_* = 0$ ($G_c = 0$), $\rho_* = 1$ (thermal problem). In this case system (2.2)-(2.4) has the nontrivial solution

$$\psi = c \equiv 0,\ \theta_1 = -(G_1 + \gamma_0\omega^2)a \exp(-lx)\ \ (a \neq 0),$$
$$\theta_2 = -(G_2 + \gamma_0\omega^2)\, a \exp(-\chi_* x/2)\, \mathrm{ch}\,[r(x+h)]\,[\mathrm{sh}\,(rh)]^{-1},$$

which exists for all values of μ satisfying the dispersion relationship

$$\mu = -(1+G_1)[r\,\mathrm{cth}\,(rh) + \chi_*/2] - G_1(l-1) - \gamma_0\varkappa_*^{-1}\omega^2[r\,\mathrm{cth}\,(rh) - \chi_*/2 + \varkappa_* l]. \tag{3.1}$$

The quantity l is the root of the equation

$$l^2 - l - \omega^2 - \mu = 0, \tag{3.2}$$

for which the condition of boundedness of the perturbations is satisfied as $x \to \infty$, i.e., $\mathrm{Re}\,l > 0$. If Q_ω denotes the function $Q_\omega(\eta, \mu) = \sqrt{\eta^2/4 + \omega^2 + \eta\mu}$, $\mathrm{Re}[Q_\omega(\eta, \mu)] > 0$, then $l = 1/2 + Q_\omega(1, \mu)$, $r = Q_\omega(\chi_*, \mu)$.

It is shown in [5] that for every complex number N for which $\mathrm{Re}\,N > 0$, the following equality is valid

$$\mathrm{sgn}[\mathrm{Im}\,(N)] = \mathrm{sgn}[\mathrm{Im}\,(N^2)] = \mathrm{sgn}[\mathrm{Im}\,(N\,\mathrm{cth}\,N)]. \tag{3.3}$$

Let us agree to use the notation $\mathrm{Re}\,N = N_1$, $\mathrm{Im}\,N = N_2$ for every complex quantity N. Using the same notation for μ and l, we can reduce (3.2) to two equivalents:

$$l_1^2 - l_2^2 - l_1 - \omega^2 - \mu_1 = 0,\quad 2l_1l_2 - l_2 - \mu_2 = 0. \tag{3.4}$$

Applying (3.3) to the complex quantities $l - 1/2$, rh ($l_1 > 1/2$, $r_1 > 0$) we obtain

$$\mathrm{sgn}\, l_2 = \mathrm{sgn}\,\{\mathrm{Im}\,[(l-1/2)^2]\} = \mathrm{sgn}\,\mu_2,$$
$$\mathrm{sgn}\,\{\mathrm{Im}\,[r\,\mathrm{cth}\,(rh)]\} = \mathrm{sgn}\,[\mathrm{Im}\,(r^2)] = \mathrm{sgn}\,\mu_2. \tag{3.5}$$

Furthermore, if $\mu_1 \geqslant -\omega^2$, then the following holds

$$\mathrm{sgn}\,(\mu_2 - l_2) = \mathrm{sgn}\, l_2 = \mathrm{sgn}\,\mu_2. \tag{3.6}$$

Indeed, if $l_2 = 0$, then $\mu_2 = 0$ by virtue of (3.5) and the validity of (3.6) is evident. For $l_2 \neq 0$ it follows from (3.4) that $(\mu_2 - l_2) = \mathrm{sgn}\,[(l_1 - 1)l_2]$, while for $\mu_1 \geqslant -\omega^2$ the quantity is $l_1 > 1$.

Let us examine the case $G_1 \geqslant 0$. Using (3.5) and comparing signs of the imaginary parts on the right and left sides of (3.1), we obtain the equality $\mathrm{sgn}\,\mu_2 = -\mathrm{sgn}\,\mu_2$, from which it follows that all the solutions of (3.1) are real. If $\mu = 0$, then $l \geqslant 1$, and the sign on the right side of (3.1) is opposite to the sign on the left side. This means that each solution of (3.1) is negative.

Let $-1 \leqslant G_1 < 0$, $\mu_1 \geqslant 0$. Converting the imaginary part of (3.1) to

$$\mu_2 - l_2 = -(1 + G_1 + \gamma_0\omega^2\varkappa_*^{-1})\,\mathrm{Im}\,[r\,\mathrm{cth}\,(rh)] - (1 + G_1 + \gamma_0\omega^2)\, l_2$$

and using (3.5), (3.6) we obtain $\mu_2 = 0$.

Assertion 1 is proved. For $G_1 \geqslant 0$ all the eigenvalues μ governed by (3.1) are real and negative, and for $-1 \leqslant G_1 < 0$ the imaginary part $\mu_2 = 0$ for every solution μ for which $\mathrm{Re}\,\mu \geqslant 0$. It hence follows that within the framework of the thermal problem the crystallization front is stable for $G_1 \geqslant 0$, while for $-1 \leqslant G_1 < 0$ the instability can be associated just with monotonic perturbations.

Setting $\mu = 0$ in (3.1), we obtain the critical dependence

$$G_1 = G_1^*(\omega^2) = \frac{r_0 \operatorname{cth}(r_0 h) + \chi_*/2 + \gamma_0 \omega^2 \varkappa_*^{-1}[r_0 \operatorname{cth}(r_0 h) - \chi_*/2 + \varkappa_* l_0]}{r_0 \operatorname{cth}(r_0 h) + \chi_*/2 + l_0 - 1}, \tag{3.7}$$

governing the stability boundary with respect to perturbations with wave number ω. Here $r_0 = \sqrt{\chi_*^2/4 + \omega^2}$, $l_0 = 1/2 + \sqrt{1/4 + \omega^2}$.

For $\gamma_0 = 0$ the value is $G_1^*(0) = -1$. As ω^2 grows, the function G_1^* increases monotonically, and for $\omega^2 \to \infty$ its value tends to $-1/2$. This means that the crystallization front is stable for $\gamma_0 = 0$ if $G_1 > -1/2$ For $G_1 = -1/2$ shortwave perturbations are most dangerous.

4. For $D_* = 0$, $\rho_* \neq 1$, we obtain a stability problem for the crystallization front of a pure melt ($c \equiv 0$, $\psi \not\equiv 0$). The dispersion equation for this problem can be reduced to the form

$$\rho_*^{-1}\mu = -(1 + G_1)[R \operatorname{cth}(Rh) + \rho_* \chi_*/2] - \gamma_0 \varkappa_*^{-1}\omega^2 [R \operatorname{cth}(Rh) - \rho_* \chi_*/2 + \varkappa_* l] - G_1(l - 1 + \varepsilon X), \tag{4.1}$$

where $X = \mu(\omega + q + l)(\omega + l)^{-1}(q + l)^{-1}$; $q = 0.5\,\mathrm{Pr}_*^{-1} + Q_\omega(\mathrm{Pr}_*, \mu)$ is the root of the equation

$$q^2 - (q - \mu)\,\mathrm{Pr}_* - \omega^2 = 0, \tag{4.2}$$

and $R = Q_\omega(\rho_* \chi_*, \rho_*^{-1}\mu)$.

If the quantities q, l (q_1, $l_1 > 0$) and μ satisfy (3.2) and (4.2), then

$$\operatorname{sgn}[\mathrm{Im}(\mu\bar{l})] = \operatorname{sgn}[\mathrm{Im}(\mu\bar{l}^2)] = \operatorname{sgn}[\mathrm{Im}(\mu\bar{q})] = \operatorname{sgn}[\mathrm{Im}(\mu\overline{lq})] = \operatorname{sgn}\mu_2. \tag{4.3}$$

We present the proof of the equality

$$\operatorname{sgn}[\mathrm{Im}(\mu\bar{l}\bar{q})] = \operatorname{sgn}\mu_2. \tag{4.4}$$

From (3.2) and (4.2) we obtain

$$\begin{aligned} \mathrm{Im}(\mu\bar{l}\bar{q}) &= |l|^2 q_2 + \omega^2 \mathrm{Im}(lq) + |l|^2 \mathrm{Im}(l\bar{q}), \\ \mathrm{Pr}_* \mathrm{Im}(\mu\bar{l}\bar{q}) &= \mathrm{Pr}_* |q|^2 l_2 + \omega^2 \mathrm{Im}(lq) - |q|^2 \mathrm{Im}(l\bar{q}). \end{aligned} \tag{4.5}$$

If $l_2 = 0$, then $\mu_2 = q_2 = 0$ and (4.4) is satisfied. If $l_2 \neq 0$, then $\operatorname{sgn}[\mathrm{Im}(l\bar{q}) = \operatorname{sgn}[l_2(q_1 - l_1 q_2/l_2)]$. In each of equalities (4.5) the sign of the first two components agrees with the sign of μ_2. If for certain μ, Pr_*, ω the inequality $q_1 \geqslant l_1 q_2/l_2$ is satisfied, then $[\mathrm{Im}(l\bar{q})] = \operatorname{sgn}\mu_2$ and the equality (4.4) follows from the first equation in (4.5), otherwise it follows from the second equation.

By using (4.3) it can be proved that if the quantities q, l (q_1, $l_1 > 0$) and μ satisfy (3.2) and (4.2) then

$$\operatorname{sgn}[\mathrm{Im}(X)] = \operatorname{sgn}\mu_2. \tag{4.6}$$

Moreover, if $\mu_i > 0$, then $\mathrm{Re}\,X > 0$.

The assertion 2 is valid. For $G_1 \geqslant 0$, $\varepsilon > 0$ all the eigenvalues μ satisfying (4.1) are real and negative, while for $G_1 \geqslant 0$, $\varepsilon < 0$ each solution μ has a negative real part. In the case $-1/2 \leqslant G_1 < 0$, $\varepsilon > -1/2$ $\mathrm{Im}\,\mu = 0$ for all the eigenvalues for which $\mathrm{Re}\,\mu \geqslant 0$.

For $G_1 \geqslant 0$, $\varepsilon > 0$ the proof (taking (4.6) into account) is analogous to the proof of assertion 1 for $G_1 \geqslant 0$.

If $G_1 \geqslant 0$, $\varepsilon < 0$, then (4.1) has no solutions with nonnegative real part since for each μ with $\operatorname{Re}\mu \geqslant 0$ the signs of the real parts of the right and left sides of this equation are opposite. For the proof it is necessary to set $\varepsilon = \rho_*^{-1} - 1$ in (4.1), use (3.2) to convert the expression $l - 1 - X$ to the form $Y\omega|\omega + l|^{-2}|q + l|^{-2}$ where

$$Y = (l-1)|q|^2\omega + (l-1)\overline{l}|q|^2 + \omega^2(\overline{q} + \overline{l}) + (2|l - 1/2|^2 + 1/2)q + (|l-1|^2 + 1)(\overline{l} - 1)q +$$

$$+ \omega[(l-1)q\overline{l} + \overline{l}|q + l|^2] + \omega^2[\overline{lq} + \overline{l}^2 + |q + l|^2],$$

and show that the real part of each component in the expression Y is nonnegative for $\mu_1 \geqslant 0$.

Let $-1/2 \leqslant G_1 < 0$. For $-1/2 < \varepsilon < 0$ the imaginary part in (4.1) can be represented in the form

$$(\varepsilon + 1/2)\mu_2 = -(\mu_2 - l_2)/2 - (G_1 + 1/2)l_2 - \gamma_0\omega^2 l_2 - (1 + G_1 + \gamma_0\varkappa_*^{-1}\omega^2)\operatorname{Im}[R\operatorname{cth}(Rh)] - \varepsilon G_1 X_2 \tag{4.7}$$

and it can be shown by using (3.3), (3.6) and (4.6) that for each μ with $\operatorname{Re}\mu \geq 0$ the signs in the left and right sides of (4.7) are opposite. In the case $\varepsilon > 0$ for the proof of assertion 2 it is necessary to convert the imaginary part of (4.1) to the form

$$\mu_2/2 = -(\mu_2 - l_2)/2 - (G_1 + 1/2)l_2 - \gamma_0\omega^2 l_2 - (1 + G_1 + \gamma_0\varkappa_*^{-1}\omega^2)\operatorname{Im}[R\operatorname{cth}(Rh)] - \varepsilon(\mu_2 - X_2/2) - \varepsilon(G_1 + 1/2)X_2$$

and to show that $\operatorname{sgn}(\mu_2 - X_2/2) = \operatorname{sgn}\mu_2$.

For $\mu = 0$ we obtain the critical dependence $G_1^*(\omega^2)$ from (4.1). This dependence is determined by (3.7) if the χ_* there is replaced by $\rho_*\chi_*$. The crystallization front is stable relative to monotonic perturbations if $G_1 > \max G_1^*(\omega^2)$. If $\gamma_0 = 0$, then $\max G_1^*(\omega^2) = G_1^*(\infty) = -1/2$ while according to assertion 2 for $G_1 > -1/2$ the oscillatory instability is impossible. Therefore, for $\gamma_0 = 0$ the stability criterion agrees with the criterion for the thermal problem.

5. In the case $\rho_* = 1$, $D_* \neq 0$ the velocity field does not experience any perturbations ($\psi \equiv 0$), and we arrive at a thermodiffusion formulation of the problem. For $\rho_* = 1$ and $h \to \infty$ the solution of the spectral problem (2.2)-(2.4) reduces to the following equation for the eigenvalues μ:

$$\mu = -G_1(l-1) - \varkappa_*^{-1}G_2(s + \chi_*) - \gamma_0\omega^2(\varkappa_*^{-1}s + l) + m_0G_c(\varkappa_*^{-1}s + l) - m_0G_cZ, \tag{5.1}$$

where $s = -\chi_*/2 + Q_\omega(\chi_*, \mu)$ and $\tau = -D_*/2 + Q_\omega(D_*, \mu)$ are the roots of the equations

$$s^2 + \chi_* s - \omega^2 - \chi_*\mu = 0, \quad \tau^2 + D_*\tau - \omega^2 - D_*\mu = 0; \tag{5.2}$$

$$Z = (\varkappa_*^{-1}s + l)(\mu + kD_*)(\tau + kD_*)^{-1}. \tag{5.3}$$

For a dilute binary alloy, $m(k-1) > 0$ follows from the Van't-Hoff equation (see [1]). Therefore $m_0G_c > 0$. Let us eliminate negative values of G_1 from consideration. For $G_1 \geqslant 0$ concentration supercooling [1, 6] characterized by the gradients $G_1 < m_0G_c$ can, however, hold in the melt.

Let $F_{(1)}$ and $F_{(2)}$ denote the functions

$$F_{(1)}(\omega^2, k, D_*, m_0G_c) = m_0G_c\tau_0(\tau_0 + kD_*)^{-1},$$
$$F_{(2)}(\omega^2, \chi_*, \varkappa_*, G_1) = [G_1(l_0 - 1) + \varkappa_*^{-1}G_2(s_0 + \chi_*)](\varkappa_*^{-1}s_0 + l_0)^{-1}, \tag{5.4}$$

where $\tau_0 = -D_*/2 + \sqrt{D_*^2/4 + \omega^2}$; $s_0 = -\chi_*/2 + \sqrt{\chi_*^2/4 + \omega^2}$.

By using (5.1), the stability condition can be found relative to the monotonic perturbations. Using the notation (5.4), we write this condition in the form

$$F_{(1)} - F_{(2)} - \gamma_0\omega^2 \begin{cases} < 0 & \text{(for all } \omega\text{): stability} \\ > 0 & \text{(for all } \omega\text{): instability} \end{cases} \tag{5.5}$$

If the quantities l, τ, s and μ (l_1, τ_1, $s_1 > 0$) satisfy (3.2) and (5.2), while Z is defined by (5.3), the the following equality is valid

$$\operatorname{sgn} Z_2 = \operatorname{sgn} s_2 = \operatorname{sgn} \mu_2. \tag{5.6}$$

Let $\gamma_0 = 0$. We prove assertion 3. If

$$\varkappa_* < 1, \quad m_0 G_c < \varkappa_* (1 - \varkappa_*)^{-1} \tag{5.7}$$

or

$$\varkappa_* > 1, \quad m_0 G_c < 1, \tag{5.8}$$

then the stability boundary is determined by the value $\mu = 0$ (stability of the process is determined by conditions (5.5)),

Let conditions (5.7) be satisfied. The assertion follows from the results of a sequential examination of certain cases: a) $G_1 \geqslant m_0 G_c$, $G_2 \geqslant m_0 G_c$. From (5.1) we find

$$\mu_2 = -(G_1 - m_0 G_c) l_2 - \varkappa_*^{-1} (G_2 - m_0 G_c) s_2 - \gamma_0 \omega^2 (\varkappa_*^{-1} s_2 + l_2) - m_0 G_c Z_2. \tag{5.9}$$

Taking account of (3.5) and (5.6), there hence follows that the eigenvalues μ are real; b) $G_2 > m_0 G_c \geqslant G_1$. We convert the imaginary part of (5.1) to the form

$$\mu_2 (1 + G_1 - m_0 G_c) = -(m_0 G_c - G_1)(\mu_2 - l_2) - \varkappa_*^{-1} (G_2 - m_0 G_c) s_2 - \gamma_0 \omega^2 (\varkappa_*^{-1} s_2 + l_2) - m_0 G_c Z_2. \tag{5.10}$$

Since $G_2 > m_0 G_c$ and $\varkappa_* \leqslant 1$, then $1 + G_1 - m_0 G_c > 0$. Taking account of (3.6) and (5.6) the realness of all the eigenvalues μ for which $\mu_1 \geqslant -\omega^2$ follows from (5.10); c) $G_1 < m_0 G_c$, $G_2 < m_0 G_c$. In this case $m_0 G_c > (G_1 + \varkappa_*^{-1} G_2)(\varkappa_*^{-1} + 1)^{-1}$, and the instability as $\omega \to \infty$ follows from (5.5); d) $G_1 > m_0 G_c \geqslant G_2$. These inequalities are incompatible if $m_0 G_c < \varkappa_* (1 - \varkappa_*)^{-1}$.

Upon compliance with conditions (5.8), it is necessary to reexamine cases "b" and "d" to prove the assertion. In case "b" it is necessary to transpose the expression $m_0 G_c l_0$ to the left side in (5.9) and to note that $(\mu_2 - m_0 G_c l_2) = \operatorname{sgn} \mu_2$, if $\mu_1 \geqslant -\omega^2$ and $m_0 G_c < 1$. Analogously, in the case "d" the realness of all the eigenvalues μ can be proved if the term $m_0 G_c \varkappa_*^{-1} S_2$ is transposed to the left side of (5.9).

The validity of the principle of the interchange of stability [7] for small values of the gradient G_c corresponding to small values of the impurity concentration $c_0 = c_\infty / k$ on the unperturbed front, is proved in the assertion. As germanium solidifies, its thermal coefficients χ and $\varkappa$ diminish. Typical for metals is the situation $\varkappa_* < 1$, $\chi_* < 1$. For the alloy Pb—Sn $k = 0.3$, $\varkappa_* = 0.54$, conditions (5.6) are satisfied if the tin concentration is $c_0 \lesssim 3.3 \cdot 10^{-3}$, while for gallium-doped germanium $k = 0.1$, $\varkappa_* = 1.6$, and conditions (5.8) are satisfied for $c_0 \leqslant 1.4 \cdot 10^{-3}$.

The problem of the stability of a plane crystallization front was examined in a thermodiffusion formulation in [1], where it was assumed that

$$\omega^2 \gg \max (1/2, \chi_*/2). \tag{5.11}$$

Taking account of (5.11), the authors of [1] neglected the convective terms in the heat-conduction equations and, starting from the stationary equations for heat and impurity transport, obtained the stability criterion ($\gamma_0 = 0$):

$$m_0 G_c (\varkappa_*^{-1} + 1) - G_1 - \varkappa_*^{-1} G_2 \begin{cases} < 0 - \text{ stability}, \\ > 0 - \text{instability}. \end{cases} \tag{5.12}$$

If the crystallization front is unstable according to the criterion (5.12), then its instability as $\omega \to \infty$ follows from (5.5). Let us mention four cases in which the equivalence of conditions (5.5) and (5.12) is shown successfully by investigating the behavior of the functions F_1 and F_2 (the stability from condition (5.5) follows from the stability by the criterion (5.12)):

1) $\chi_* = 1$, $\varkappa_* \leqslant 1$;
2) $1/3 \leqslant \chi_* < 1$, $D_* > 1$, $k \geqslant 1$;
3) $1/3 \leqslant \chi_* < 1$, $1/2 < k < 1$, $D_* > (2k-1)^{-1}$;
4) $D_* \gg 1$, $k^2 D_* \gg 1$.

Characteristic for metals are the values $D_* \geqslant 10^4$ and the last condition can be considered satisfied for $k \geqslant 10^{-1}$.

Let us note that criterion (5.12) can be obtained if the dispersion relation (5.1) is replaced by the approximate equation

$$\mu = -G_1 l_m - \varkappa_*^{-1} G_2 s_m + m_0 G_c (\varkappa_*^{-1} s_m + l_m) - m_0 G_c Z_m \tag{5.13}$$

and the validity of the principle of interchange of the stability is taken. In (5.13) $l_m = \sqrt{\omega^2 + \mu}$, $s_m = \sqrt{\omega^2 + \chi_* \mu}$ (l_{m1}, $s_{m1} > 0$) and Z_m is obtained from Z by replacing s and l by s_m and l_m.

The following assertion is valid: For $\varkappa_* \geqslant 1$, $\chi_* \geqslant 1$ and for $\varkappa_* < 1$, $\chi_* \leqslant 1$, $G_1 \geqslant G_2$ each solution μ of (5.13) is real, while for $\varkappa_* < 1$, $G_1 < G_2$ the stability boundary is determined by the value $\mu = 0$.

Let $\rho_* \neq 1$, $D_* \neq 0$ (general case). If the parameter χ_* is replaced by $\rho_* \chi_*$ in the function $F_{(2)}$, then the stability conditions relative to monotonic perturbations will be determined, as before, by using (5.5). The conditions for the equivalence of 1-4 remain valid if the replacement of χ_* by $\rho_* \chi_*$ is made.

The problem of small perturbations (2.2)-(2.4) does not contain the acceleration of gravity g. For absolutely incompressible media gravity does not influence the stability of the crystallization front. The convective instability of the crystallization process was studied in [8].

The author is grateful to V. V. Pukhnachev for constant attention to the research.

LITERATURE CITED

1. W. Mullins and R. Sekerka, "Stability of the plane interphasial surface in the crystallization of a dilute binary alloy," in: Problems of Crystal Growth [Russian translation], Mir, Moscow (1964).
2. L. I. Sedov, Mechanics of a Continuous Medium [in Russian], Vol. 1, Nauka, Moscow (1973).
3. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Oxford Univ. Press (1959).
4. L. I. Rubinshtein, Stefan Problem [in Russian], Zvaigzne, Riga (1967).
5. B. D. Turland and R. S. Peckover, "The stability of planar melting fronts in two-phase thermal Stefan problems," J. Inst. Math. Appl., 25, No. 1 (1980).
6. R. F. Sekerka, "Morphological stability," in: Crystal Growth: An Introduction, North-Holland, Amsterdam (1973).
7. C. C. Lin, Theory of Hydrodynamic Stability [Russian translation], IL, Moscow (1958).
8. S. R. Coriell, M. R. Cordes, et al., "Convective and interfacial instabilities during unidirectional solidification of a binary alloy," J. Crystal Growth, 49, No. 1 (1980).